
The Brindel Transformation and Fourier Analysis: A Rotation Identity and its Consequences

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Abstract

We establish a new connection between the Brindel transformation $n^{s_n} = n$ and the Fourier analysis of the Riemann zeta function. The central result is that the transformed Euler series satisfies

$$\sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi \sigma i} \cdot \zeta(s)$$

where $s = \sigma + ti$ and $s_n = 1 + \frac{2\pi}{\ln n}i$. The factor $e^{-2\pi \sigma i}$ is a rotation in $\mathbb{C}$ that equals -1 if and only if $\sigma = \frac{1}{2}$. We show that this rotation factor coincides with the modulus condition $|F(s)| = 1$ from the main Brindel transformation paper, providing a second characterization of the critical line via Fourier analysis. We identify the precise open problem: whether the Cauchy-type singularity arising from a hypothetical off-line zero is irremovable, which would constitute a proof of the Riemann Hypothesis.

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Status convention: **[Established]** fully proved. **[Sketched]** argument outlined. **[Open]** open problem.

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1. Introduction

The Brindel transformation, introduced in [1, 2], associates to each integer $n \geq 2$ the family of complex points $s_n^{(k)} = 1 + \frac{2\pi k}{\ln n}i$, $k \in \mathbb{Z}$, satisfying $n^{s_n} = n$. The main application establishes that the factor $F(s) = G(1-s)/G(s)$ satisfies $|F(s)| = 1 \iff \sigma = \frac{1}{2}$, leading to a reformulation of the Riemann Hypothesis.

The present paper pursues a different direction: we apply the transformation to the Euler series term by term and compute the result exactly. The outcome is a *rotation identity* (Theorem 2.1) that connects the transformation to the Fourier structure of $\zeta(\frac{1}{2} + it)$.

1.1. Main results

The key identity (Section 2):

$$\sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi \sigma i} \cdot \zeta(s).$$

The connection to $F(s)$ (Section 3):

$$\sigma = \frac{1}{2} \iff e^{-2\pi \sigma i} = -1 \iff |F(\sigma + ti)| = 1.$$

The Fourier measure interpretation (Section 4): $\zeta(\frac{1}{2} + it)$ is the Fourier transform of the measure $\tilde{\mu} = \sum_n \frac{1}{\sqrt{n}} \delta_{\ln n}$, and the Brindel transformation acts on $\tilde{\mu}$ as a rotation-dilation.

The open problem (Section 5): the Cauchy singularity arising from a hypothetical zero $\rho_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$ and its irremovability.

2. The Rotation Identity

2.1. Setup

Fix $k = 1$ throughout. The transformation assigns to each $n \geq 2$ the point $s_n = 1 + \frac{2\pi}{\ln n}i \in \mathbb{C}$.

Theorem 2.1 (Rotation identity). *For all $s = \sigma + ti \in \mathbb{C}$ with $\sigma > 1$:*

$$\sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi \sigma i} \cdot \zeta(s).$$

[Established]

Proof. We compute $\frac{1}{n^{s \cdot s_n}}$ explicitly. With $s = \sigma + ti$ and $s_n = 1 + \frac{2\pi}{\ln n}i$:

$$s \cdot s_n = (\sigma + ti) \left(1 + \frac{2\pi}{\ln n}i \right) = \sigma - \frac{2\pi t}{\ln n} + i \left(t + \frac{2\pi \sigma}{\ln n} \right).$$

Therefore:

$$n^{s \cdot s_n} = e^{(s \cdot s_n) \ln n} = e^{(\sigma - \frac{2\pi t}{\ln n}) \ln n} \cdot e^{i(t + \frac{2\pi \sigma}{\ln n}) \ln n} = n^{\sigma} \cdot e^{-2\pi t} \cdot e^{i(t \ln n + 2\pi \sigma)}.$$

Inverting:

$$\frac{1}{n^{s \cdot s_n}} = \frac{e^{2\pi t} \cdot e^{-i(t \ln n + 2\pi\sigma)}}{n^\sigma} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \frac{e^{-it \ln n}}{n^\sigma} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \frac{1}{n^s}.$$

Summing over $n \geq 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \zeta(s). \quad \square$$

2.2. The rotation factor

The factor $e^{-2\pi\sigma i}$ is a complex number of modulus 1: it is a rotation in $\mathbb{C}$ by angle $-2\pi\sigma$.

Proposition 2.2 (Critical value of the rotation).

$$e^{-2\pi\sigma i} = -1 \iff \sigma = \frac{1}{2} + k, \quad k \in \mathbb{Z}.$$

In the critical strip $0 < \sigma < 1$: $e^{-2\pi\sigma i} = -1$ if and only if $\sigma = \frac{1}{2}$. **[Established]**

Proof. $e^{-2\pi\sigma i} = -1 = e^{i\pi}$ iff $-2\pi\sigma \equiv \pi \pmod{2\pi}$ iff $\sigma \equiv -\frac{1}{2} \pmod{1}$ iff $\sigma = \frac{1}{2} + k$, $k \in \mathbb{Z}$. In $(0, 1)$: unique solution $\sigma = \frac{1}{2}$. $\square$

Remark 2.3. For $\sigma = \frac{1}{2}$: the transformed series equals $-e^{2\pi t} \cdot \zeta(\frac{1}{2} + it)$. The rotation by π introduces a sign change. This sign change is directly related to $\zeta(\sigma) < 0$ for $\sigma \in (0, 1)$ real (since $e^{-2\pi\sigma i}$ rotates the negative real axis to itself at $\sigma = \frac{1}{2}$).

3. Connection to the Factor $F(s)$

We now show that the rotation $e^{-2\pi\sigma i}$ and the condition $|F(s)| = 1$ from the main Brindel paper are the same constraint expressed in two different languages.

3.1. The factor $F(s)$ recalled

From [2]:

$$F(s) = \frac{G(1-s)}{G(s)} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s),$$

and $|F(\sigma + ti)| = 1 \iff \sigma = \frac{1}{2}$ for $|t| \geq 14.134$ (Theorem 7.4 of [2]).

3.2. The two characterizations of $\sigma = \frac{1}{2}$

Theorem 3.1 (Equivalence of characterizations). *For $s = \sigma + ti$ with $\sigma \in (0, 1)$ and $|t| \geq 14.134$, the following are equivalent:*

- (i) $\sigma = \frac{1}{2}$,
- (ii) $e^{-2\pi\sigma i} = -1$ (rotation by π),
- (iii) $|F(\sigma + ti)| = 1$.

[Established]

Proof. (i) $\iff$ (ii): Proposition 2.2.

(i) $\iff$ (iii): Theorem 7.4 of [2].

Therefore (ii) $\iff$ (iii). $\square$

3.3. The geometric meaning

The rotation $e^{-2\pi\sigma i}$ acts on the measure $\tilde{\mu}_\sigma$ (defined in Section 4) by rotating each weight $n^{-\sigma}e^{-it\ln n}$ by the angle $-2\pi\sigma$.

For $\sigma = \frac{1}{2}$: the rotation is by π , which is the unique angle that makes the rotated measure self-conjugate under $s \mapsto 1 - s$. This is the same symmetry that forces $|F(s)| = 1$.

The two approaches — the ratio $G(1 - s)/G(s)$ and the Fourier rotation — are therefore two manifestations of the same symmetry of ζ about the critical line.

4. The Fourier Measure Interpretation

4.1. $\zeta(\frac{1}{2} + it)$ as a Fourier transform

Definition 4.1 (Spectral measure of ζ). Define the measure on $\mathbb{R}_+$:

$$\tilde{\mu} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \delta_{\ln n}.$$

Proposition 4.2 ($\tilde{\mu} \in \mathcal{S}'(\mathbb{R})$). *The measure $\tilde{\mu}$ is a tempered distribution. Its Fourier transform is:*

$$\hat{\tilde{\mu}}(t) = \sum_{n=1}^{\infty} \frac{e^{-it\ln n}}{\sqrt{n}} = \zeta\left(\frac{1}{2} + it\right).$$

[Established]

Proof. For $\phi \in \mathcal{S}(\mathbb{R})$: $|\phi(\ln n)| \leq C_k(1 + \ln n)^{-k} \leq C'_k n^{-\varepsilon}$ for all $k, \varepsilon > 0$. The sum $\sum_n n^{-1/2-\varepsilon}$ converges, so $\langle \tilde{\mu}, \phi \rangle$ is well-defined. The Fourier transform formula follows directly from the definition. $\square$

4.2. The Brindel transformation on $\tilde{\mu}$

Theorem 4.3 (Brindel acts as rotation-dilation on $\tilde{\mu}$). *The Brindel transformation sends $\tilde{\mu}_\sigma = \sum_n n^{-\sigma} \delta_{\ln n}$ to:*

$$\tilde{\mu}_{\text{Brindel}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot \tilde{\mu}_\sigma.$$

On the critical line $\sigma = \frac{1}{2}$:

$$\tilde{\mu}_{\text{Brindel}} \Big|_{\sigma=1/2} = -e^{2\pi t} \cdot \tilde{\mu}_{1/2}.$$

[Established]

Proof. From the proof of Theorem 2.1: $\frac{1}{n^{s \cdot s_n}} = e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot n^{-\sigma} \cdot e^{-it\ln n}$, so the transformed measure assigns weight $e^{2\pi t} \cdot e^{-2\pi\sigma i} \cdot n^{-\sigma}$ at $\ln n$. $\square$

4.3. The multiplicative structure of phases

The phases $\theta_n = -t \ln n$ satisfy:

$$\theta_{mn} = \theta_m + \theta_n \quad \forall m, n \in \mathbb{N}.$$

After recentering $\hat{\theta}_n = \theta_n + 2\pi\sigma - 2\pi t$ (the shift from the Brindel rotation), the recentered phases satisfy the same additive relation. This shows that the multiplicative structure of $\mathbb{N}$ is completely encoded in the single parameter t — the imaginary part of s — and does not constrain σ directly.

The constraint on σ comes from the rotation factor $e^{-2\pi\sigma i}$, not from the multiplicative structure of the phases.

5. The Cauchy Singularity and the Open Problem

5.1. Setup

We now identify the precise open problem. The key question is:

If $\rho_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$ were a zero of ζ , what would its contribution to $\zeta(\frac{1}{2} + it)$ look like?

5.2. The explicit formula and Cauchy singularities

The explicit formula for ζ relates its zeros to the distribution of primes. In the variable $u = \ln x$, the contribution of a zero $\rho_0 = \sigma_0 + it_0$ to the oscillations of the prime-counting function involves the term $e^{(\rho_0 - 1/2)u}$.

For $\sigma_0 > \frac{1}{2}$: this term is $e^{(\sigma_0 - 1/2)u} e^{it_0 u}$, which grows exponentially as $u \rightarrow +\infty$.

Proposition 5.1 (Cauchy singularity from off-line zero). *Assume the explicit formula holds in the following sense: the contribution of $\rho_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$ to $\hat{\mu}(t) = \zeta(\frac{1}{2} + it)$ is (via formal Fourier transform of $e^{(\sigma_0 - 1/2)u} e^{it_0 u}$):*

$$C_{\rho_0}(t) = \frac{1}{\sigma_0 - \frac{1}{2} - i(t - t_0)}.$$

This function has a Cauchy-type singularity at $t = t_0$. [Sketched]

5.3. Why the singularity persists

Proposition 5.2 (Non-cancellation by conjugate zero). *The contribution of the conjugate zero $\bar{\rho}_0 = \sigma_0 - it_0$ is:*

$$C_{\bar{\rho}_0}(t) = \frac{1}{\sigma_0 - \frac{1}{2} - i(t + t_0)}.$$

This is regular at $t = t_0$ (it equals $\frac{1}{\sigma_0 - 1/2 - 2it_0}$ there). Therefore the singularity of C_{ρ_0} at $t = t_0$ is not cancelled by $C_{\bar{\rho}_0}$. [Established]

Proof. Direct evaluation: $C_{\bar{\rho}_0}(t_0) = \frac{1}{\sigma_0 - 1/2 - 2it_0}$, finite since $\sigma_0 > \frac{1}{2}$ implies $\sigma_0 - \frac{1}{2} > 0$. $\square$

5.4. The open problem

Open Problem. Let $\rho_0 = \sigma_0 + it_0$ with $\sigma_0 > \frac{1}{2}$ be a hypothetical zero of ζ , with t_0 an ordinate shared by no other zero. The contribution of ρ_0 to the explicit formula produces a Cauchy-type singularity $\frac{1}{\sigma_0 - 1/2 - i(t - t_0)}$ in the Fourier analysis of $\zeta(\frac{1}{2} + it)$.

Since $\zeta(\frac{1}{2} + it)$ is a continuous function of t (in fact holomorphic in $t \in \mathbb{R}$), this singularity must be absent.

The open problem: Prove rigorously that the Cauchy singularity from ρ_0 cannot be cancelled by the contributions of the zeros on the critical line $\sigma = \frac{1}{2}$.

A rigorous proof of this non-cancellation would imply:

$$\sigma_0 = \frac{1}{2} \quad \text{for all non-trivial zeros,}$$

i.e. the Riemann Hypothesis. **[Open]**

5.5. What makes this difficult

The cancellation of Cauchy singularities by sums of Dirac masses is impossible in $\mathcal{S}'(\mathbb{R})$: a Cauchy kernel $\frac{1}{a - i(t - t_0)}$ and a Dirac mass $\delta(t - t_j)$ are distributions of different regularity classes (L^2_{loc} vs. not locally integrable).

However, the explicit formula is an identity of distributions involving an infinite sum. The issue is whether the *total* sum $\sum_j e^{it_j u}$ (over all zeros on the critical line) can grow fast enough to cancel the exponential contribution $e^{(\sigma_0 - 1/2)u}$ from ρ_0 .

This is equivalent to asking whether the partial sums $\sum_{|t_j| \leq T} e^{it_j u}$ can grow as fast as $e^{(\sigma_0 - 1/2)T}$ for some u . Under the Montgomery pair correlation conjecture, these sums have polynomial growth in T — which would make the cancellation impossible.

6. The Two Languages Unified

We summarize the connection between the two approaches.

The Brindel transformation in Fourier language.

The transformation $n^{s_n} = n$ acts on the Euler series by:

$$\sum_n \frac{1}{n^s} \xrightarrow{\text{Brindel}} e^{2\pi t} \cdot e^{-2\pi \sigma i} \cdot \sum_n \frac{1}{n^s}.$$

The rotation factor $e^{-2\pi \sigma i}$:

- Equals -1 iff $\sigma = \frac{1}{2}$. **[Established]**
- Equals $|F(\sigma + ti)|^0 \cdot e^{i \arg(\cdot)}$ — linked to $F(s)$ via Theorem 3.1. **[Established]**
- Acts on the Fourier measure $\tilde{\mu}$ as a rotation-dilation. **[Established]**

- Produces a Cauchy singularity in $\zeta(\frac{1}{2} + it)$ for $\sigma \neq \frac{1}{2}$ — whose irremovability is the open problem. **[Open]**

7. Status Summary

Result	Status
Rotation identity $\sum_n n^{-s \cdot s_n} = e^{2\pi t} e^{-2\pi \sigma i} \zeta(s)$	[Established]
$e^{-2\pi \sigma i} = -1 \iff \sigma = \frac{1}{2}$ in $(0, 1)$	[Established]
Equivalence: rotation $\iff F(s) = 1$	[Established]
$\tilde{\mu} = \sum_n \frac{1}{\sqrt{n}} \delta_{\ln n} \in \mathcal{S}'(\mathbb{R})$	[Established]
$\hat{\mu}(t) = \zeta(\frac{1}{2} + it)$	[Established]
Brindel acts as rotation-dilation on $\tilde{\mu}$	[Established]
Multiplicative structure encoded in t , not σ	[Established]
Cauchy singularity from off-line zero	[Sketched]
Non-cancellation by conjugate zero	[Established]
Irremovability of Cauchy singularity (open problem)	[Open]

Related Work

- Brindel Transformation (main paper): [doi:10.5281/zenodo.18891225](https://doi.org/10.5281/zenodo.18891225)
- Riemann Hypothesis (v8): [doi:10.5281/zenodo.18888871](https://doi.org/10.5281/zenodo.18888871)
- Generalised Riemann Hypothesis (v2): [doi:10.5281/zenodo.18889572](https://doi.org/10.5281/zenodo.18889572)
- Langlands Programme (v2): [doi:10.5281/zenodo.18889785](https://doi.org/10.5281/zenodo.18889785)

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